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# Separation of variables solutions of nonlinear reaction–diffusion systems

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**Abstract**
Sign invariance plays an important role in the study of an explicit solution to single partial differential equations. In this paper, we extend the sign invariant theory to study the nonlinear reaction–diffusion system. As a consequence, we obtain some new explicit solutions to the resulting systems, and we illustrate the exact solutions for a nonlinear parabolic system.

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## 1. Introduction

Nonlinear reaction–diffusion systems (RDs) have been widely studied over the past decades. These systems arise naturally as models of the evolution problem in the real world [1–6]. In this paper, we consider the nonlinear reaction–diffusion systems (RDs)

$$u_t = (f(u, v)u_x)_x + g(u, v) \qquad v_t = (p(u, v)v_x)_x + q(u, v), \tag{1.1}$$

where $u = u(x, t)$ and $v = v(x, t)$ are unknown differentiable functions, the subscripts $t$ and $x$ to the functions denote differentiation with respect to these variables, and $f(u, v)$, $g(u, v)$, $p(u, v)$ and $q(u, v)$ are smooth functions to be determined later. There are many papers devoted to the investigation of existence and uniqueness problems, asymptotic behaviour of solutions and so on ([6, 7] and the references cited therein). On the other hand, there are many approaches to find exact solutions of RDs, such as the Lie symmetry [8–12], the ansatz-based method [12, 13], the Galilei-invariant method [14], the Painlevé analysis [15] and the algebraic method [16]. In these papers, many interesting results had been obtained.

It is well known that, for a single equation, there are many theories and approaches for finding its exact solutions, such as the Lie symmetry, non-classical method, sign-invariant theory and so on. In these methods, sign-invariant theory plays an important role for finding the explicit solutions of the single-diffusion equation. The sign-invariant theory was introduce originally by Galaktionov [17–19], which was extended by the maximum principle for second-order linear parabolic equations, the main idea, as follows [17].

Consider a single reaction–diffusion equation

$$u_t = (f(u)u_x)_x + g(u).$$

We assume that the solution set $\{u(x,t)\}$ to the given equation is not empty. Next, we introduce a general quadratic Hamilton–Jacobi (HJ) operator of the form

$$\mathcal{H}(u) = u_t - H(u) \equiv u_t - [\alpha(u)(u_x)^2 + \beta(u)u_x + h(u)],$$

where the functions $\alpha(u)$, $\beta(u)$ and $h(u)$ depend on the coefficients of the RD equation.

**Definition 1.** *We say that the operator HJ is the sign invariant of the RD equation if it preserves both the signs,* $\geqslant 0$ *and* $\leqslant 0$*, on the solution* $u(x,t)$ *to the RD equation. It means that*

$$\begin{aligned} &\mathcal{H}(u) \geqslant 0\,(\leqslant 0) \qquad \textit{in} \quad \mathbf{R} \qquad \textit{for} \quad t = 0 \\ &\Longrightarrow \mathcal{H}(u) \geqslant 0\,(\leqslant 0) \qquad \textit{in} \quad \mathbf{R} \qquad \textit{for} \quad t > 0. \end{aligned} \tag{1.2}$$

**Remark 1.** It is important to note that under assumption (1.2) the operator HJ is also a zero invariant of the RD equation, i.e., there holds

$$\begin{aligned} &\mathcal{H}(u) = 0 \qquad \text{in} \quad \mathbf{R} \qquad \text{for} \quad t = 0 \\ &\Longrightarrow \mathcal{H}(u) = 0 \qquad \text{in} \quad \mathbf{R} \qquad \text{for} \quad t > 0. \end{aligned} \tag{1.3}$$

Indeed, we can obtain the exact solutions for the corresponding equation by utilizing the sign-invariant theory. In [18], the author utilized the sign-invariant theory to study maximal sign invariants describing all possible sign invariants of a prescribe structure, and in [19], the author considered the general first-order sign invariant for quasilinear heat equations. As a consequence, they constructed some new exact solutions of resulting equations.

In the present paper, for finding the exact solutions of the nonlinear reaction–diffusion system, we extend the sign-invariant theory by introducing the HJ system as follows:

$$\mathcal{H}_1(u,v) = u_t - \xi(u,v), \qquad \mathcal{H}_2(u,v) = v_t - \eta(u,v), \tag{1.4}$$

where $\xi(u,v)$ and $\eta(u,v)$ are some smooth functions depending on the coefficients of the RDs (1.1).

Now, we summarize the main results of the present paper. In section 2, the sign invariant is applied to the RDs; the main results of this section are describe in theorem 2.1. In section 3, we arrive at many separable solutions for the considered system. Section 4 contains concluding remarks on this work.

## 2. Sign invariant of system (1.1)

In this section, we are looking for a sign invariant for system (1.1) of the form

$$\mathcal{H}_1(u,v) = u_t - \xi(u,v) \qquad \mathcal{H}_2(u,v) = v_t - \eta(u,v), \tag{2.1}$$

where the smooth functions $\xi(u,v)$, $\eta(u,v)$ depend on the coefficients of the system and determined later.

We now state the first main result of the present paper.

**Theorem 2.1.** *System (2.1) is a sign invariant of RDs (1.1) if the coefficients $f(u,v)$, $g(u,v)$, $p(u,v)$, $q(u,v)$, $\xi(u,v)$ and $\eta(u,v)$ satisfy the following system of partial differential equations:*

$$\begin{aligned}
&A_1 \equiv \xi f_{uu} + \eta f_{uv} + \xi_u f_u + \eta_u f_v - \xi \frac{f_u^2}{f} - \eta \frac{f_u f_v}{f} + \xi_{uu} f = 0,\\
&A_2 \equiv 2\xi_v f_u + \xi f_{uv} + \eta f_{vv} + \eta_v f_v - \eta \frac{f_v^2}{f} - \xi \frac{f_u f_v}{f} + 2\xi_{uv} f - \xi_v \frac{p_u f}{p} = 0,\\
&A_3 \equiv \xi_v f_v + \xi_{vv} f - \xi_v \frac{f p_v}{p} = 0,\\
&A_4 \equiv \xi \frac{f_u(\xi - g)}{f} + \eta \frac{f_v(\xi - g)}{f} + \xi_v \frac{f(\eta - q)}{p} - \xi_u g \xi g_u + \eta g_v - \xi_v \eta = 0,\\
&A_5 \equiv \xi_v p_u + \xi p_{uv} + \eta p_{vv} + \eta_v p_v - \frac{\xi p_u p_v}{p} - \frac{\eta p_v^2}{p} + \eta_{vv} p = 0,\\
&A_6 \equiv \xi p_{uu} + \eta p_{uv} + \xi_u p_u + 2\eta_u p_v - \frac{\eta p_u p_v}{p} + 2\eta_{uv} p - \frac{\eta_u f_v p}{f} = 0,\\
&A_7 \equiv \eta_u p_u + \eta_{uu} p - \eta_u \frac{f_u p}{f} = 0,\\
&A_8 \equiv \xi \frac{p_u(\eta - q)}{p} + \eta \frac{p_v(\eta - q)}{p} + \eta_u \frac{p(\xi - g)}{f} - \eta_v q + \xi g_u + \eta q_v - \xi \eta_u = 0.
\end{aligned} \tag{2.2}$$

**Proof.** We set

$$J_1 = \mathcal{H}_1 = u_t - \xi(u,v), \qquad J_2 = \mathcal{H}_2 = v_t - \eta(u,v). \tag{2.3}$$

Then by differentiating $J_1$ and $J_2$ with respect to $t$, we arrive at

$$J_{1t} = \mathcal{H}_{1t} = u_{tt} - \xi_u u_t - \xi_v v_t, \qquad J_{2t} = \mathcal{H}_{2t} = v_{tt} - \eta_u u_t - \eta_v v_t, \tag{2.4}$$

and substituting the second derivative $u_{tt}$, $v_{tt}$ from the RDs differentiating in $t$ and calculating other lower order ones $v_{xt}$, $v_t$, $v_{xxt}$, $u_{xt}$, $u_t$ and $u_{xxt}$ from system (2.3) into (2.4), we arrive at two equations of the form

$$A_1 u_x^2 + A_2 u_x v_x + A_3 v_x^2 + A_4 = 0, \qquad A_5 v_x^2 + A_6 u_x v_x + A_7 u_x^2 + A_8 = 0. \tag{2.5}$$

Then vanishing of the expression system (2.5) leads to the determining equations (2.2). □

According to the proof, to get the exact solutions of system (1.1), we must solve $f(u,v)$, $g(u,v)$, $p(u,v)$, $q(u,v)$, $\xi(u,v)$ and $\eta(u,v)$ from the determining equations (2.2). For the general case, the equations dependent on $f(u,v)$, $g(u,v)$, $p(u,v)$, $q(u,v)$, $\xi(u,v)$ and $\eta(u,v)$ are complicated; it seems very hard to write down the solutions explicitly. So, as usual, we assume the following two models which are being widely studied over the past decades.

*Case 1*. $f = p = 1$, $g \neq 0$, $q \neq 0$.

*Case 2*. $f = p \neq 1$, $g = q = 0$.

As a consequence, the following statements are valid.

**Theorem 2.2.** *System (2.1) is a sign invariant of the RDs*

$$u_t = u_{xx} + g(u,v), \qquad v_t = v_{xx} + q(u,v),$$

*if the functions* $g(u, v), q(u, v), \xi(u, v)$*, and* $\eta(u, v)$ *satisfy the following system:*

$$\begin{aligned} &\xi(u, v) = a_1 u + a_2 v + a_3, \\ &\eta(u, v) = b_1 u + b_2 v + b_3, \\ &\xi g_u + \eta g_v - \xi_u g - \xi_v q = 0, \\ &\xi q_u + \eta q_v - \eta_v q - \eta_u g = 0. \end{aligned} \tag{2.6}$$

It is interesting to note that the last two equations in (2.6) can be written in the form of two coupled evolution equations:

$$\left(\xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} - \xi_u\right) g = \xi_v q, \qquad \left(\xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} - \eta_v\right) g = \eta_v g.$$

This means that the family of curves $\phi(u, v) =$ const. is invariant under the operator $\xi \partial_u + \eta \partial_v$.

**Theorem 2.3.** *System (2.1) is a sign invariant of the RDs*

$$u_t = (f(u, v) u_x)_x, \qquad v_t = (f(u, v) v_x)_x,$$

*if the functions* $f(u, v), \xi(u, v)$ *and* $\eta(u, v)$ *satisfy the following system:*

$$\begin{aligned} &\xi(u, v) = a_1 u + a_2 v + a_3, \\ &\eta(u, v) = b_1 u + b_2 v + b_3, \\ &\xi f_u + \eta f_v = 0. \end{aligned} \tag{2.7}$$

*Hereafter,* $a_i, b_i$ *and* $c_i, i = 1, 2, 3$*, are arbitrary constants.*

## 3. Exact solutions for cases 1 and 2

In this section, we present a full description of sign invariant and the corresponding explicit solutions in cases 1 and 2.

*Case 3.1.* In this case, system (1.1) becomes a nonlinear system of evolution equations

$$u_t = u_{xx} + g(u, v), \qquad v_t = v_{xx} + q(u, v).$$

Some methods can be used to consider the system [8–12], and many solutions had been obtained. To illustrate the notion of theorem 2.2, we have $\xi(u, v), \eta(u, v)$:

$$\xi(u, v) = a_1 u + a_2 v + a_3, \qquad \eta(u, v) = b_1 u + b_2 v + b_3. \tag{3.1}$$

For simplicity, we set $a_3 = b_3 = 0$; if $a_3 \neq 0, b_3 \neq 0$, we can make a translation

$$\begin{cases} u = \tilde{u} + c_1 \\ v = \tilde{v} + c_2 \end{cases}$$

to obtain $a_3 = b_3 = 0$. Thus the determining equations can be formulated as follows:

$$\begin{aligned} &\xi(u, v) = a_1 u + a_2 v, && \eta(u, v) = b_1 u + b_2 v, \\ &\xi g_u + \eta g_v - \xi_u g - \xi_v q = 0, && \xi q_u + \eta q_v - \eta_v q - \eta_u g = 0. \end{aligned} \tag{3.2}$$

**Remark 2.** It is easy to see that $g(u, v), q(u, v)$ are determined by the two-dimension dynamical, and we can solve $u, v$ from $\xi(u, v), \eta(u, v)$. But in this case, the solution which is obtained by $\xi(u, v)$ and $\eta(u, v)$ depends on two arbitrary functions $\Phi_1(x), \Phi_2(x)$. So if we want to construct the exact solutions, we need to determine $\Phi_1(x), \Phi_2(x)$ with $g(u, v), q(u, v)$

by utilizing the compatibility conditions. In this paper *the compatibility conditions* means that $u$, $v$, $g(u, v)$ and $q(u, v)$ fulfil the original system.

We assert that the resulting several cases are distinguished.

*Subcase 3.1.1.* $a_1 \neq 0, a_2 = 0$. In this subcase $\xi = a_1 u$ and $\eta = b_1 u + b_2 v$.

It is easy to have

$$\begin{aligned}
&g(u, v) = G_1(\alpha)u,\\
&q(u, v) = u^{\frac{b_2}{a_1}} G_1(\alpha) - \frac{b_1}{b_2} G_2(\alpha),\\
&\alpha = v u^{-\frac{b_2}{a_1}} - \frac{b_1}{(a_1 - b_2)} u^{1-\frac{b_2}{a_1}},\\
&u(x, t) = \mathrm{e}^{a_1 t}\Phi_1(x),\\
&v(x, t) = \frac{b_1 \Phi_2(x)}{a_1 - b_2}\, \mathrm{e}^{a_1 t}\Phi_1(x) + \Phi_2(x)\,\mathrm{e}^{b_2 t},
\end{aligned}$$

for $a_1 \neq b_2$. Here $\Phi_1(x)$ and $\Phi_2(x)$ are smooth functions with $x$, and $\Phi_1(x)$, $\Phi_2(x)$, $G_1(\alpha)$ and $G_2(\alpha)$ fulfil the following constraints:

$$\begin{aligned}
&G_1(\alpha) = 0,\\
&\Phi_2''(x) - b_2\Phi_2(x) + (\Phi_1(x))^{\frac{b_2}{a_1}} G_2(\alpha) = 0,\\
&\Phi_1(x) = C_1\,\mathrm{e}^{\sqrt{a_1}x} + C_2\,\mathrm{e}^{-\sqrt{a_1}x} && \text{for} \quad a_1 > 0,\\
&\Phi_1(x) = C_1 \cos\sqrt{a_1}x + C_2 \sin\sqrt{a_1}x && \text{for} \quad a_1 < 0.
\end{aligned}$$

If $a_1 = b_2$, we obtain

$$\begin{aligned}
&u(x, t) = \mathrm{e}^{a_1 t}\Phi_1(x),\\
&v(x, t) = (b_1\Phi_1(x) + \Phi_2(x))\,\mathrm{e}^{a_1 t},\\
&g(u, v) = G_1(\alpha)u,\\
&q(u, v) = \frac{b_1}{a_1} G_1(\alpha) v \ln u + G_2(\alpha) v,\\
&\alpha = \frac{a_1 v - b_1 u \ln u}{a_1 u}.
\end{aligned}$$

It is easy to calculate that $\Phi_1(x)$, $\Phi_2(x)$, $G_1(\alpha)$ and $G_2(\alpha)$ satisfy the following equations:

$$\begin{aligned}
&\Phi_1'' + (G_1 - a_1)\Phi_1 = 0,\\
&\Phi_2'' - a_1\Phi_2 - b_1\Phi_1 + \frac{b_1}{a_1} G_1 \ln\Phi_1 + G_2 = 0.
\end{aligned}$$

*Subcase 3.1.2.* $a_1 = 0, a_2 \neq 0$. In this subcase $\xi = a_2 v$ and $\eta = b_1 u + b_2 v$. We have the solutions as follows:

(i) $\Delta = b_2^2 + 4a_2 b_1 > 0$.

$$\begin{aligned}
&u(x, t) = \Phi_1(x)\,\mathrm{e}^{\delta_1 t} + \Phi_2(x)\,\mathrm{e}^{\delta_2 t},\\
&v(x, t) = \frac{1}{a_2}(\Phi_1(x)\delta_1\,\mathrm{e}^{\delta_1 t} + \Phi_2(x)\delta_2\,\mathrm{e}^{\delta_2 t})
\end{aligned}$$

where $\delta_1 = (b_2 + \sqrt{\Delta})/2$, $\delta_2 = (b_2 - \sqrt{\Delta})/2$.

(ii) $\Delta = b_2^2 + 4a_2 b_1 = 0$.

$$\begin{aligned}
&u(x, t) = (\Phi_1(x) + \Phi_2(x)t)\,\mathrm{e}^{\delta t},\\
&v(x, t) = \frac{1}{a_2}(\delta\Phi_1(x) + \Phi_2(x) + \Phi_2(x)t\delta)\,\mathrm{e}^{\delta t}
\end{aligned}$$

where $\delta = b_2/2$.

(iii) $\Delta = b_2^2 + 4a_2b_1 < 0$.

$$u(x,t) = (\Phi_1(x)\cos\delta_1 t + \Phi_2(x)\sin\delta_2 t)\,\mathrm{e}^{\delta_1 t},$$
$$v(x,t) = \frac{1}{a_2}[(\Phi_1(x)\delta_1 + \Phi_2\delta_2)\cos\delta_1 t + (\Phi_2(x)\delta_1 - \Phi_1(x)\delta_2)\sin\delta_2 t)]\,\mathrm{e}^{\delta_1 t}$$

where $\delta_1 = b_2/2$, $\delta_2 = (\sqrt{-\Delta})/2$.

In this case, $g(u,v)$ and $q(u,v)$ satisfy

$$a_2^2v^2g_{vv} + 2a_2v(b_1u + b_2v)g_{uv} + (b_1u + b_2v)^2g_{vv} + a_2b_1(vg_v + ug_u - g) = 0,$$
$$q = \frac{1}{a_2}[a_2vg_u + (b_1u + b_2v)g_v],$$

and $\Phi_1(x)$, $\Phi_2(x)$, $g(u,v)$, $q(u,v)$ fulfil the compatibility conditions.

In particular, when $a_1 = b_2 = 0, a_2b_1 > 0$, the sign invariants are $\xi(u,v) = a_2v$ and $\eta(u,v) = b_1u$. Solving the sign invariants, we arrive at

$$g(u,v) = \left(G_1(\alpha) - \frac{G_2(\alpha)}{a_2^2}\right)(\sqrt{a_2b_1}u - a_2v),$$
$$q(u,v) = \left(G_1(\alpha) + \frac{G_2(\alpha)}{a_2^2}\right)(\sqrt{a_2b_1}v - b_1u),$$
$$\alpha = v^2 - \frac{b_1}{a_2}u^2,$$
$$u(x,t) = \mathrm{e}^{\sqrt{a_2b_1}t}\phi_1(x) + \mathrm{e}^{-\sqrt{a_2b_1}t}\phi_2(x),$$
$$v(x,t) = \sqrt{\frac{b_1}{a_2}}\left(\mathrm{e}^{\sqrt{a_2b_1}t}\phi_1(x) - \mathrm{e}^{-\sqrt{a_2b_1}t}\phi_2(x)\right),$$

where $\phi_1(x)$, $\phi_2(x)$, $G_1(\alpha)$ and $G_2(\alpha)$ fulfil

$$\phi_1'' - \sqrt{a_2b_1}\phi_1 = 0, \qquad \phi_2'' - 2G_1\sqrt{a_2b_1}\phi_2 = 0, \qquad G_2 = \tfrac{1}{2}a_2^2.$$

**Example 1.** When

$$g(u,v) = 0$$
$$q(u,v) = -\frac{b_1}{b_2}u^{-\frac{b_2}{a_1}}\left(v - \frac{1}{a_1 - b_2}u\right)$$

we obtain the exact solution for nonlinear RDs

$$\begin{cases} u_t = u_{xx}, \\ v_t = v_{xx} - \dfrac{b_1}{b_2}u^{-\frac{b_2}{a_1}}\left(v - \dfrac{1}{a_1 - b_2}u\right), \end{cases}$$

as follows:

$$\begin{aligned}
&u(x,t) = \mathrm{e}^{a_1 t}\Phi_1(x), \\
&v(x,t) = \tfrac{b_1\Phi_2(x)}{a_1 - b_2}\,\mathrm{e}^{a_1 t}\Phi_1(x) + \Phi_2(x)\,\mathrm{e}^{b_2 t}, \\
&\Phi_1(x) = C_1\,\mathrm{e}^{\sqrt{a_1}x} + C_2\,\mathrm{e}^{-\sqrt{a_1}x} && \text{for} \quad a_1 > 0, \\
&\Phi_1(x) = C_1\cos\sqrt{a_1}x + C_2\sin\sqrt{a_1}x && \text{for} \quad a_1 < 0, \\
&\Phi_2(x) = C_3\,\mathrm{e}^{\sqrt{b_2 - 1}x} + C_4\,\mathrm{e}^{-\sqrt{b_2 - 1}x} && \text{for} \quad b_2 > 1, \\
&\Phi_2(x) = C_3\cos\sqrt{1 - b_2}x + C_4\sin\sqrt{1 - b_2}x && \text{for} \quad b_2 < 1.
\end{aligned}$$

**Example 2.** When

$$g(u,v) = \beta_1u^{1+\alpha_1}v^{-\alpha_1}, \qquad q(u,v) = \beta_2u^{\alpha_2}v^{1-\alpha_2},$$

the RDs are the biological pattern model arising in hydra [20],

$$u_t = u_{xx} + \beta_1 u^{1+\alpha_1} v^{-\alpha_1}, \qquad v_t = v_{xx} + \beta_2 u^{\alpha_2} v^{1-\alpha_2}.$$

It is invariant with respect to the Galilei algebra $AG(1,1)$ [11]. By a straightforward calculation we obtain that

$$\begin{cases} \xi(u,v) = a_1 u, \\ \eta(u,v) = b_2 v, \end{cases}$$

is sign invariant of the system, and the explicit solutions are as follows:

$$u(x,t) = \mathrm{e}^{a_1 t}\Phi_1(x), \qquad v(x,t) = \mathrm{e}^{b_2 t}\Phi_2(x),$$

where $\Phi_1(x)$ and $\Phi_2(x)$ fulfil the constraints:

$$\Phi_1'' + \left[\beta_1\left(\frac{\Phi_1}{\Phi_2}\right)^{\alpha_1} - a_1\right]\Phi_1 = 0, \qquad \Phi_2'' + \left[\beta_2\left(\frac{\Phi_1}{\Phi_2}\right)^{\alpha_2} - b_2\right]\Phi_2 = 0.$$

*Case 3.2.* In this case, the standard nonlinear heat equations

$$u_t = (f(u,v)u_x)_x, \qquad v_t = (f(u,v)v_x)_x, \tag{3.3}$$

follow from the system (1.1). On the basis of theorem 2.3, the determining equations (2.2) are as follows:

$$\begin{aligned} &\xi(u,v) = a_1 u + a_2 v + a_3, \\ &\eta(u,v) = b_1 u + b_2 v + b_3, \\ &\xi f_u + \eta f_v = 0. \end{aligned} \tag{3.4}$$

Similar to case 3.1, we set $a_3 = b_3 = 0$ and consider the following subcases.

*Subcase 3.2.1.* $a_1 \neq 0, a_2 = 0$. In this subcase $\xi(u,v) = a_1 u$ and $\eta(u,v) = b_1 u + b_2 v$.
Solving (3.4), we have

(i) $a_1 \neq b_2$:

$$f(u,v) = F(\alpha), \qquad \alpha = \frac{a_1 v - b_1 u - b_2 v}{(a_1 - b_2)u^{\frac{b_2}{a_1}}},$$

$$u(x,t) = \mathrm{e}^{a_1 t}\Phi_1(x), \qquad v(x,t) = \frac{b_1\Phi_1(x)}{a_1 - b_2}\,\mathrm{e}^{a_1 t}\Phi_2(x) + \Phi_2(x)\,\mathrm{e}^{b_2 t},$$

where $\Phi_1(x)$, $\Phi_2(x)$ and $F(\alpha)$ satisfy

$$F\Phi_1'' + F_\alpha\left(\Phi_2'\Phi_1^{-\frac{b_2}{a_1}} - \frac{b_2}{a_1}\Phi_2\Phi_1^{-\frac{b_2-a_1}{a_1}}\Phi_1'\right)\Phi_1' - a_1\Phi_1 = 0,$$

$$F\Phi_2'' + F_\alpha\left(\Phi_2'\Phi_1^{-\frac{b_2}{a_1}} - \frac{b_2}{a_1}\Phi_2\Phi_1^{-\frac{b_2-a_1}{a_1}}\Phi_1'\right)\Phi_2' - b_2\Phi_2 = 0.$$

(ii) $a_1 = b_2$:

$$f(u,v) = F(\alpha), \qquad \alpha = \frac{a_1 v - b_1 u \ln u}{a_1 u},$$

$$u(x,t) = \mathrm{e}^{a_1 t}\Phi_1(x), \qquad v(x,t) = (b_1\Phi_1(x)t + \Phi_2(x))\,\mathrm{e}^{a_1 t},$$

where $\Phi_1(x)$, $\Phi_2(x)$ and $F(\alpha)$ satisfy the following identities:

$$F\Phi_1'' + F_\beta\left(\frac{\Phi_2'\Phi_1 - \Phi_1'\Phi_2}{\Phi_1^2} + \frac{b_1}{a_1\Phi_1}\right)\Phi_1' - a_1\Phi_1 = 0,$$

$$F\Phi_2'' + F_\beta\left(\frac{\Phi_2'\Phi_1 - \Phi_1'\Phi_2}{\Phi_1^2} + \frac{b_1}{a_1\Phi_1}\right)\Phi_2' - a_1\Phi_2 = 0.$$

*Subcase 3.2.2.* $a_1 = 0$. In this case, $\xi(u, v) = a_2 v$ and $\eta(u, v) = b_1 u + b_2 v$. We have the solutions of (3.3) as follows.

(i) $\Delta = b_2^2 + 4a_2 b_1 > 0$:

$$f(u, v) = F(\alpha),$$
$$\alpha = -\frac{1}{2}\ln\left|\frac{b_1}{a_2} + \frac{b_2 v}{a_2 u} - \frac{v^2}{u^2}\right| + \frac{a_2}{\sqrt{\Delta}}\ln\left|\frac{-2a_2 v + b_2 u - \sqrt{\Delta}u}{-2a_2 v + b_2 u + \sqrt{\Delta}u}\right| + c,$$
$$u(x, t) = \Phi_1(x)\,\mathrm{e}^{\delta_1 t} + \Phi_2(x)\,\mathrm{e}^{\delta_2 t},$$
$$v(x, t) = \frac{1}{a_2}(\Phi_1(x)\,\mathrm{e}^{\delta_1 t} + \Phi_2(x)\,\mathrm{e}^{\delta_2 t}),$$

where $\delta_1 = (b_2 + \sqrt{\Delta})/2$, $\delta_1 = (b_2 - \sqrt{\Delta})/2$.

(ii) $\Delta = 0$:

$$f(u, v) = F(\alpha), \qquad \alpha = \frac{b_2 u}{2a_v - b_2 u} - \ln\left|\frac{v}{u} - \frac{b_2}{2a_2}\right| + c,$$
$$u(x, t) = (\Phi_1(x) + \Phi(x)t)\,\mathrm{e}^{\delta t},$$
$$v(x, t) = \frac{1}{a_2}(\delta\Phi_1(x) + \Phi_2(x) + \Phi_2(x)\delta t)\,\mathrm{e}^{\delta t}.$$

where $\delta = b_2/2$.

(iii) $\Delta < 0$:

$$f(u, v) = F(\alpha),$$
$$\alpha = -\frac{1}{2}\ln\left|\frac{b_1}{a_2} + \frac{b_2 v}{a_2 u} - \frac{v^2}{u^2}\right| + \frac{b_2}{\sqrt{\Delta}}\arctan\left(\frac{b_2 u - 2a_2 v}{u\sqrt{\Delta}}\right) + c,$$
$$u(x, t) = (\Phi_1(x)\cos\beta t + \Phi_2(x)\sin\beta t)\,\mathrm{e}^{\gamma t},$$
$$v(x, t) = \frac{1}{a_2}((\Phi_1(x)\gamma + \Phi_2(x)\beta)\cos\beta t + (\Phi_2(x)\gamma - \Phi_1(x)\beta)\sin\beta t)\,\mathrm{e}^{\gamma t}.$$

where $\gamma = b_2/2$, $\beta = \sqrt{-b_2^2 - 4a_2 b_1}/2$.

In subcase 3.2.2, $\Phi_1(x)$, $\Phi_2(x)$ and $F(\alpha)$ fulfil the compatibility conditions.

In particular, when $a_1 = b_1 = 0, a_2 b_2 \neq 0$, the sign invariants are $\xi(u, v) = a_2 v$ and $\eta(u, v) = b_2 v$.

Solving the sign invariant, we obtain

$$f(u, v) = F(\alpha), \qquad \alpha = \frac{u^2}{a_2} - \frac{v^2}{b_1},$$
$$u(x, t) = \frac{a_2}{b_2}\,\mathrm{e}^{b_2 t}\Phi_1(x) + \Phi_2(x), \qquad v(x, t) = \mathrm{e}^{b_2 t}\Phi_1(x),$$

where $F(\alpha)$, $\Phi_1(x)$ and $\Phi_2(x)$ satisfy

$$F\Phi_2'' - F_\alpha \Phi_2' = 0, \qquad b_2\Phi_1 + F_\alpha \Phi_1'\Phi_2' - F\Phi_1'' = 0.$$

## 4. Concluding remarks

This paper deals with the application of sign-invariant theory to nonlinear RDs, and we get many separation of variables solutions of two types. For the general case, we also get the exact solutions for the given HJ system $\{\xi(u, v), \eta(u, v)\}$. This method is the same as the generalized conditional symmetry method [21–29]. And we can extend the HJ system

$$\mathcal{H}_1(u, v) = u_t - \alpha(u) + \beta(v), \qquad \mathcal{H}_2(u, v) = v_t - \gamma(u) + \eta(v),$$

to obtain the explicit solutions. On the other hand, for giving RDs, we can also construct exact solutions by sign-invariant theory.

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